Chapter 2 : Basic Structures: Sets, Functions, Sequences, Sums, and Matrices

Lecture 9 : Exercises & Correction

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Exercise 9

Determine whether each of these functions is a bijection from **R** to **R**.

a) $f(x) = 2x + 1$
b) $f(x) = x^2 + 1$
c) $f(x) = x^3$
d) $f(x) = (x^2 + 1)/(x^2 + 2)$

Exercise 10

Suppose that g is a function from A to B and f is a function from B to C.

a) Show that if both f and g are one-to-one functions, then $f \circ g$ is also one-to-one.

b) Show that if both f and g are onto functions, then $f \circ g$ is also onto.

Solution Exercise 9

a) Yes; All real numbers can be doubled and increased by one to equal a unique real number.

b) No; This combination completely leaves out all negative numbers and zero, not to mention it is not one to one.

c) Yes; All real numbers can be cubed to equal unique real numbers.

d) No; Though the domain includes all real numbers, the numerator and denominators both output positive numbers individually, so all outputs must be positive. (Graphing the function also reveals that it is not at all one-to-one.)

Solution Exercise 10

SOLUTION

(a) Given: $g : A \to B$ is one-to-one and $f : B \to C$ is one-to-one.

To proof: $f \circ g$ is one-to-one

PROOF

f is one-to-one: if $f(x) = f(y)$, then $x = y$
g is one-to-one: if $g(x) = g(y)$, then $x = y$

Let us assume $(f \circ g)(a) = (f \circ g)(b)$. Use the definition of composition:

$$f(g(a)) = f(g(b))$$

Since f is one-to-one:

$$g(a) = g(b)$$

Since g is one-to-one:

$$a = b$$

By the definition of one-to-one, we have then shown that $f \circ g$ is one-to-one.
□

DEFINITIONS

The function f is **one-to-one** if and only if $f(a) = f(b)$ implies that $a = b$ for all a and b in the domain.

The function f is **onto** if and only if for every element $b \in B$ there exist an element $a \in A$ such that $f(a) = b$.

Composition of f and g: $(f \circ g)(a) = f(g(a))$

(b) Given: $g : A \to B$ is onto and $f : B \to C$ is onto.

To proof: $f \circ g$ is onto.

PROOF

f is onto: $\forall c \in C \exists b \in B : f(b) = c$
g is onto: $\forall b \in B \exists a \in A : g(a) = b$

Let $x \in C$. Since f is onto, there then exists an element $y \in B$ such that:

$$f(y) = x$$

Since g is onto, there then exists an element $z \in A$ such that:

$$g(z) = y$$

The composition evaluated at z is then:

$$(f \circ g)(z) = f(g(z)) = f(y) = x$$

Thus for every element c in C, there then exists an element a in A such that $(f \circ g)(a) = c$. By the definition of onto, we have then shown that $f \circ g$ is onto.
$\square$

Exercise 11

Find $f \circ g$ and $g \circ f$, where $f(x) = x^2 + 1$ and $g(x) = x + 2$, are functions from $\mathbf{R}$ to $\mathbf{R}$.

Exercise 12

Let $f(x) = ax + b$ and $g(x) = cx + d$, where a, b, c, and d are constants. Determine necessary and sufficient conditions on the constants a, b, c, and d so that $f \circ g = g \circ f$.

Solution Exercise 11

DEFINITIONS

Composition of f and g: $(f \circ g)(a) = f(g(a))$

SOLUTION

Given: $g : \mathbf{R} \to \mathbf{R}$ and $f : \mathbf{R} \to \mathbf{R}$.

$$f(x) = x^2 + 1$$

$$g(x) = x + 2$$

Since f and g are both functions from $\mathbf{R}$ to $\mathbf{R}$, $f \circ g$ and $g \circ f$ are also functions from $\mathbf{R}$ to $\mathbf{R}$

Use the definition of composition:

$$(f \circ g)(x) = f(g(x)) = f(x+2) = (x+2)^2 + 1 = x^2 + 4x + 5$$
$$(g \circ f)(x) = g(f(x)) = g(x^2+1) = (x^2+1) + 2 = x^2 + 3$$

Solution Exercise 12

SOLUTION

Given:

$$f(x) = ax + b$$

$$g(x) = cx + d$$

Use the definition of composition:

$$(f \circ g)(x) = f(g(x)) = f(cx + d) = a(cx + d) + b = acx + ad + b$$

$$(g \circ f)(x) = g(f(x)) = g(ax + b) = c(ax + b) + d = acx + bc + d$$

The two compositions have to be equal ($f \circ g = g \circ f$).

$$acx + ad + b = acx + bc + d$$

Subtract acx from each side of the previous equation:

$$ad + b = bc + d$$

Thus the necessary and sufficient conditions on the constants is: $ad + b = bc + d$

Exercise 13

Let f be a function from the set A to the set B. Let S and T be subsets of A. Show that

a) $f(S \cup T) = f(S) \cup f(T)$.

b) $f(S \cap T) \subseteq f(S) \cap f(T)$.

Exercise 14

Let f be the function from **R** to **R** defined by $f(x) = x^2$. Find

a) $f^{-1}(\{1\})$.

b) $f^{-1}(\{x \mid 0 < x < 1\})$.

c) $f^{-1}(\{x \mid x > 4\})$.

Solution Exercise 13

SOLUTION

(a) Given: $f : A \to B$ and $S \subseteq A$ and $T \subseteq A$

To proof: $f(S \cup T) = f(S) \cup f(T)$

PROOF

FIRST PART Let $x \in f(S \cup T)$. Then there exists an element $y \in S \cup T$ such that $f(y) = x$. By the definition of the union:

$$y \in S \vee y \in T$$

$f(S)$ contains all elements that are the image of an element in S.
$f(T)$ contains all elements that are the image of an element in T.

$$f(y) \in f(S) \vee f(y) \in f(T)$$

Since $f(y) = x$

$$x \in f(S) \vee x \in f(T)$$

By the definition of union:

$$x \in f(S) \cup f(T)$$

By the definition of subset: $f(S \cup T) \subseteq f(S) \cup f(T)$.

SECOND PART Let $x \in f(S) \cup f(T)$. By the definition of the union:

$$x \in f(S) \vee x \in f(T)$$

Then there exists an element $y \in S$ or $y \in T$ such that $f(y) = x$.

$$y \in S \vee y \in T$$

By the definition of union:

$$y \in S \cup T$$

$f(S \cup T)$ contains all elements that are the image of an element in $S \cup T$.

$$f(y) \in f(S \cup T)$$

Since $f(y) = x$

$$x \in f(S \cup T)$$

By the definition of subset: $f(S) \cup f(T) \subseteq f(S \cup T)$.

CONCLUSION Since $f(S) \cup f(T) \subseteq f(S \cup T)$ and $f(S \cup T) \subseteq f(S) \cup f(T)$, the two sets have to be equal:

$$f(S) \cup f(T) = f(S \cup T)$$

□

(b) Given: $f : A \to B$ and $S \subseteq A$ and $T \subseteq A$

To proof: $f(S \cap T) \subseteq f(S) \cap f(T)$

PROOF

Let $x \in f(S \cap T)$. Then there exists an element $y \in S \cap T$ such that $f(y) = x$. By the definition of the intersection:

$$y \in S \wedge y \in T$$

$f(S)$ contains all elements that are the image of an element in S.
$f(T)$ contains all elements that are the image of an element in T.

$$f(y) \in f(S) \wedge f(y) \in f(T)$$

Since $f(y) = x$

$$x \in f(S) \wedge x \in f(T)$$

By the definition of intersection:

$$x \in f(S) \cap f(T)$$

By the definition of subset: $f(S \cap T) \subseteq f(S) \cap f(T)$.
□

Solution Exercise 14

Given: $f : \mathbf{R} \to \mathbf{R}$

$$f(x) = x^2$$

(a) $f^{-1}(\{1\})$ contains all elements of $\mathbf{R}$ that has as image 1.

$$f(x) = 1$$

Since $f(x) = x^2$

$$x^2 = 1$$

Take the square root of each side of the equation (note: the square root can be negative or positive).

$$x = \pm\sqrt{1} = \pm 1$$

$f^{-1}(1)$ thus contains -1 and 1.

$$f^{-1}(1) = \{-1, 1\}$$

(b) $f^{-1}(\{x|0 < x < 1\})$ contains all elements of $\mathbf{R}$ that has as image a real number between 0 and 1.

$$0 < f(x) < 1$$

Since $f(x) = x^2$

$$0 < x^2 < 1$$

The square x^2 is less than 1 if x is between -1 and 1.
The square x^2 is more than 0, when x is not equal to 0.

$$(-1 < x < 1) \wedge (x \neq 0)$$

$f^{-1}(\{x|0 < x < 1\})$ thus contains all real numbers between -1 and 1, except for 0.

$$f^{-1}(\{x|0 < x < 1\}) = \{x|(-1 < x < 1) \wedge (x \neq 0)\}$$

Or equivalently: $f^{-1}(\{x|0 < x < 1\}) = \{x|(-1 < x < 0) \vee (0 < x < 1)\}$

(c) $f^{-1}(\{x|x > 4\})$ contains all elements of $\mathbf{R}$ that has as image a real number greater than 4.

$$f(x) > 4$$

Since $f(x) = x^2$

$$x^2 > 4$$

The square x^2 is more than 4 if x is more than 2 or x is less than -2 (since $2^2 = 4 = (-2)^2$).

$$(x < -2) \vee (x > 2)$$

$f^{-1}(\{x|x > 4\})$ thus contains all real numbers smaller than -2 and larger than 2.

$$f^{-1}(\{x|x > 4\}) = \{x|(x < -2) \vee (x > 2)\}$$

Exercise 15

Find the Boolean product of **A** and **B**, where

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} \quad \text{and} \quad \mathbf{B} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \\ 1 & 0 \end{bmatrix}.$$

Exercise 16

Let

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad \mathbf{B} = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix}.$$

Find

a) $\mathbf{A} \vee \mathbf{B}$. **b)** $\mathbf{A} \wedge \mathbf{B}$. **c)** $\mathbf{A} \odot \mathbf{B}$.

Solution Exercise 15

$$A \odot B = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} \odot \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \\ 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} (1 \wedge 1) \vee (0 \wedge 0) \vee (0 \wedge 1) \vee (1 \wedge 1) & (1 \wedge 0) \vee (0 \wedge 1) \vee (0 \wedge 1) \vee (1 \wedge 0) \\ (0 \wedge 1) \vee (1 \wedge 0) \vee (0 \wedge 1) \vee (1 \wedge 1) & (0 \wedge 0) \vee (1 \wedge 1) \vee (0 \wedge 1) \vee (1 \wedge 0) \\ (1 \wedge 1) \vee (1 \wedge 0) \vee (1 \wedge 1) \vee (1 \wedge 1) & (1 \wedge 0) \vee (1 \wedge 1) \vee (1 \wedge 1) \vee (1 \wedge 0) \end{bmatrix}$$

$$= \begin{bmatrix} 1 \vee 0 \vee 0 \vee 1 & 0 \vee 0 \vee 0 \vee 0 \\ 0 \vee 0 \vee 0 \vee 1 & 0 \vee 1 \vee 0 \vee 0 \\ 1 \vee 0 \vee 1 \vee 1 & 0 \vee 1 \vee 1 \vee 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 1 \end{bmatrix}$$

Solution Exercise 16

Given:

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\mathbf{B} = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix}$$

(a) $\mathbf{A} \vee \mathbf{B}$ takes the disjunction of each element of $\mathbf{A}$ with the corresponding element (same row and same column) of $\mathbf{B}$.

$$\mathbf{A} \vee \mathbf{B} = \begin{bmatrix} 1 \vee 0 & 0 \vee 1 & 1 \vee 1 \\ 1 \vee 1 & 1 \vee 0 & 0 \vee 1 \\ 0 \vee 1 & 0 \vee 0 & 1 \vee 1 \end{bmatrix}$$

$b_1 \vee b_2$ is equal to 1 if $b_1 = 1$ or $b_2 = 1$ (else $b_1 \vee b_2$ is equal to 0).

$$= \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$$

(c) The ijth element of the Boolean product $\mathbf{A} \odot \mathbf{B}$ takes the disjunction of all conjunctions of corresponding elements $\mathbf{A}$ and $\mathbf{B}$ in the ith row of $\mathbf{A}$ and in the jth column of $\mathbf{B}$.

$$\mathbf{A} \odot \mathbf{B} = \begin{bmatrix} (1 \wedge 0) \vee (0 \wedge 1) \vee (1 \wedge 1) & (1 \wedge 1) \vee (0 \wedge 0) \vee (1 \wedge 0) & (1 \wedge 1) \vee (0 \wedge 1) \vee (1 \wedge 1) \\ (1 \wedge 0) \vee (1 \wedge 1) \vee (0 \wedge 1) & (1 \wedge 1) \vee (1 \wedge 0) \vee (0 \wedge 0) & (1 \wedge 1) \vee (1 \wedge 1) \vee (0 \wedge 1) \\ (0 \wedge 0) \vee (0 \wedge 1) \vee (1 \wedge 1) & (0 \wedge 1) \vee (0 \wedge 0) \vee (1 \wedge 0) & (0 \wedge 1) \vee (0 \wedge 1) \vee (1 \wedge 1) \end{bmatrix}$$

$b_1 \wedge b_2$ is equal to 1 if $b_1 = b_2 = 1$(else $b_1 \wedge b_2$ is equal to 0).

$$= \begin{bmatrix} 0 \vee 0 \vee 1 & 1 \vee 0 \vee 0 & 1 \vee 0 \vee 1 \\ 0 \vee 1 \vee 0 & 1 \vee 0 \vee 0 & 1 \vee 1 \vee 0 \\ 0 \vee 0 \vee 1 & 0 \vee 0 \vee 0 & 0 \vee 0 \vee 1 \end{bmatrix}$$

$b_1 \vee b_2 \vee b_3$ is equal to 1 if $b_1 = 1$ or $b_2 = 1$ or $b_3 = 1$ (else $b_1 \vee b_2 \vee b_3$ is equal to 0).

$$= \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$$

(b) $\mathbf{A} \wedge \mathbf{B}$ takes the conjunction of each element of $\mathbf{A}$ with the corresponding element (same row and same column) of $\mathbf{B}$.

$$\mathbf{A} \wedge \mathbf{B} = \begin{bmatrix} 1 \wedge 0 & 0 \wedge 1 & 1 \wedge 1 \\ 1 \wedge 1 & 1 \wedge 0 & 0 \wedge 1 \\ 0 \wedge 1 & 0 \wedge 0 & 1 \wedge 1 \end{bmatrix}$$

$b_1 \wedge b_2$ is equal to 1 if $b_1 = b_2 = 1$(else $b_1 \wedge b_2$ is equal to 0).

$$= \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Exercise 17

Let

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}.$$

Find

a) $\mathbf{A}^{[2]}$. **b)** $\mathbf{A}^{[3]}$.

c) $\mathbf{A} \vee \mathbf{A}^{[2]} \vee \mathbf{A}^{[3]}$.

Given:

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

(a) $\mathbf{A}^{[2]}$ is the boolean product of $\mathbf{A}$ with itself, thus $\mathbf{A} \odot \mathbf{A}$.

The ijth element of the Boolean product $\mathbf{A} \odot \mathbf{B}$ takes the disjunction of all conjunctions of corresponding elements $\mathbf{A}$ and $\mathbf{B}$ in the ith row of $\mathbf{A}$ and in the jth column of $\mathbf{B}$.

$$\mathbf{A}^{[2]} = \mathbf{A} \odot \mathbf{A}$$

$$= \begin{bmatrix} (1 \wedge 1) \vee (0 \wedge 1) \vee (0 \wedge 0) & (1 \wedge 0) \vee (0 \wedge 0) \vee (0 \wedge 1) & (1 \wedge 0) \vee (0 \wedge 1) \vee (0 \wedge 0) \\ (1 \wedge 1) \vee (0 \wedge 1) \vee (1 \wedge 0) & (1 \wedge 0) \vee (0 \wedge 0) \vee (1 \wedge 1) & (1 \wedge 0) \vee (0 \wedge 1) \vee (1 \wedge 0) \\ (0 \wedge 1) \vee (1 \wedge 1) \vee (0 \wedge 0) & (0 \wedge 0) \vee (1 \wedge 0) \vee (0 \wedge 1) & (0 \wedge 0) \vee (1 \wedge 1) \vee (0 \wedge 0) \end{bmatrix}$$

$b_1 \wedge b_2$ is equal to 1 if $b_1 = b_2 = 1$(else $b_1 \wedge b_2$ is equal to 0).

$$= \begin{bmatrix} 1 \vee 0 \vee 0 & 0 \vee 0 \vee 0 & 0 \vee 0 \vee 0 \\ 1 \vee 0 \vee 0 & 0 \vee 0 \vee 1 & 0 \vee 0 \vee 0 \\ 0 \vee 1 \vee 0 & 0 \vee 0 \vee 0 & 0 \vee 1 \vee 0 \end{bmatrix}$$

$b_1 \vee b_2 \vee b_3$ is equal to 1 if $b_1 = 1$ or $b_2 = 1$ or $b_3 = 1$ (else $b_1 \vee b_2 \vee b_3$ is equal to 0).

$$= \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

(b)$\mathbf{A}^{[3]}$ is the boolean product of $\mathbf{A}^{[2]}$ with $\mathbf{A}$, thus $\mathbf{A}^{[2]} \odot \mathbf{A}$.

The ijth element of the Boolean product $\mathbf{A} \odot \mathbf{B}$ takes the disjunction of all conjunctions of corresponding elements $\mathbf{A}$ and $\mathbf{B}$ in the ith row of $\mathbf{A}$ and in the jth column of $\mathbf{B}$.

$$\mathbf{A}^{[3]} = \mathbf{A}^{[2]} \odot \mathbf{A}$$

$$= \begin{bmatrix} (1 \wedge 1) \vee (0 \wedge 1) \vee (0 \wedge 0) & (1 \wedge 0) \vee (0 \wedge 0) \vee (0 \wedge 1) & (1 \wedge 0) \vee (0 \wedge 1) \vee (0 \wedge 0) \\ (1 \wedge 1) \vee (1 \wedge 1) \vee (0 \wedge 0) & (1 \wedge 0) \vee (1 \wedge 0) \vee (0 \wedge 1) & (1 \wedge 0) \vee (1 \wedge 1) \vee (0 \wedge 0) \\ (1 \wedge 1) \vee (0 \wedge 1) \vee (1 \wedge 0) & (1 \wedge 0) \vee (0 \wedge 0) \vee (1 \wedge 1) & (1 \wedge 0) \vee (0 \wedge 1) \vee (1 \wedge 0) \end{bmatrix}$$

$b_1 \wedge b_2$ is equal to 1 if $b_1 = b_2 = 1$(else $b_1 \wedge b_2$ is equal to 0).

$$= \begin{bmatrix} 1 \vee 0 \vee 0 & 0 \vee 0 \vee 0 & 0 \vee 0 \vee 0 \\ 1 \vee 1 \vee 0 & 0 \vee 0 \vee 0 & 0 \vee 1 \vee 0 \\ 1 \vee 0 \vee 0 & 0 \vee 0 \vee 1 & 0 \vee 0 \vee 0 \end{bmatrix}$$

$b_1 \vee b_2 \vee b_3$ is equal to 1 if $b_1 = 1$ or $b_2 = 1$ or $b_3 = 1$ (else $b_1 \vee b_2 \vee b_3$ is equal to 0).

$$= \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

(c) $\mathbf{A} \vee \mathbf{B}$ takes the disjunction of each element of $\mathbf{A}$ with the corresponding element (same row and same column) of $\mathbf{B}$.

$$\mathbf{A} \vee \mathbf{A}^{[2]} \vee \mathbf{A}^{[3]} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \vee \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \vee \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \vee 1 \vee 1 & 0 \vee 0 \vee 0 & 0 \vee 0 \vee 0 \\ 1 \vee 1 \vee 1 & 0 \vee 1 \vee 0 & 1 \vee 0 \vee 1 \\ 0 \vee 1 \vee 1 & 1 \vee 0 \vee 1 & 0 \vee 1 \vee 0 \end{bmatrix}$$

$b_1 \vee b_2$ is equal to 1 if $b_1 = 1$ or $b_2 = 1$ (else $b_1 \vee b_2$ is equal to 0).

$$= \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$